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ON THE CALCULUS OF FUNCTIONS DERIVED FROM LIMITING-RATIOS.*

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I. INTRODUCTION.

1. The object of this essay is an investigation of the foundation of the Calculus of Limiting-Ratios as a particular case of the Calculus of Finite Differences considered in a generalized sense. It possesses the scientific interest of showing that the Differential Calculus of Newton is but one particular case of an infinite number of such Calculi, and is the simplest of them all.

The limiting-ratios of the Calculus as founded by Newton and Leibnitz, or as they are usually called the successive derivatives of $f(x)$, are defined to be

$$f^{m+1}(x) = \mathop{\pounds}_{h=0} \frac{f^m(x+h) - f^m(x)}{h},$$

$m = 0, 1, 2, \ldots$, giving the successive derivatives. These functions are derived, each from the preceding, through exactly the same operation, which is therefore essentially a repetitive operation. The Calculus of Derivatives is a repetitive calculus whose operator is (d/dx), and the repetitive character is symbolized by the operative index following the exponential law

$$\left[\frac{d}{dx}\right]\left[\frac{d^n}{dx^n}\right] = \left[\frac{d^{n+1}}{dx^{n+1}}\right].$$

In like manner, the Calculus of Finite Differences, as ordinarily considered, is also a repetitive calculus. The successive functions of this Calculus are defined to be

$$\Delta_h^m f(x),$$

wherein, if

$$E_h^m f(x) = f(x + mh),$$

then

$$\Delta_h^m f(x) = (E_h - 1)^m f(x),$$

using the notations usually employed in the Calculi of Finite Differences and Enlargement.

The Differential Calculus can be otherwise founded as a particular case of the Calculus of Finite Differences, and the successive derivatives defined to be the functions

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* Read before The American Mathematical Society in New York, May, 1895.

$$f^m(x) = \mathop{\mathcal{L}}_{\Delta x = 0} \frac{\Delta_h^m f(x)}{(\Delta x)^m} = \mathop{\mathcal{L}}_{h=0} \frac{\Delta_h^m f(x)}{h^m}. \qquad (m = 1, 2, 3, \ldots)$$

The repetitive operator (Δ_h^m/h^m) furnishing what is called the successive difference-ratios of the function, and

$$\frac{\Delta}{h}\left[\frac{\Delta^n}{h^n}\right] = \left[\frac{\Delta^{n+1}}{h^{n+1}}\right].$$

2. It is to be observed, in the Calculus of Finite Differences, that the interval between the arguments is constantly equal to h, whence all successive differences of these arguments vanish after the first. It is now proposed to generalize the Calculus of Finite Differences and to naturalize the Calculi of Limiting-Ratios as particular cases of this Calculus. Throughout this paper, $f(x)$, unless otherwise specifically stated, will be taken to represent a uniform, finite, and continuous function of x which is expressible by Taylor's series throughout a certain definite interval, say, from $x = a$ to $x = c$.

We define ε to be an operator whose function it is to act on the variable as follows:

$$\varepsilon_h^r \cdot f(x_g) = f(x_{g+rh}).$$

We define δ to be an operator which produces the effect

$$\begin{aligned} \delta_h^n \cdot f(x_g) &= (1 - \varepsilon_h)^n \cdot f(x_g), \\ &= f(x_g) - C_{n,1} f(x_{g+h}) + C_{n,2} f(x_{g+2h}) - \ldots + (-1)^n f(x_{g+nh}). \end{aligned}$$

Thus the operations have the relations

$$\delta_h = 1 - \varepsilon_h, \quad \varepsilon_h = 1 - \delta_h.$$

The corresponding nth difference of the argument is

$$\begin{aligned} \delta_h^n \cdot x_g &= (1 - \varepsilon_h)^n \cdot x_g \\ &= x_g - C_{n,1} x_{g+h} + C_{n,2} x_{g+2h} - \ldots + (-1)^n x_{g+nh}. \end{aligned}$$

We shall be concerned with the limiting value of the ratio

$$\mathop{\mathcal{L}}_{h=0} \frac{\delta_h^n f(x_g)}{\delta_h^n x_g},$$

which is the nth generalized-difference-ratio of the nth generalized difference of the function to a corresponding nth difference of the variable. We call the above function the nth difference-ratio-limit, and at the risk of being guilty of surreptitious coinage, we abbreviate this into nth *differell.*

A symbolic notation for the operation will be required, and we write the nth differell of $f(x)$

$$\left[\frac{d^n}{d^n x}\right] f(x) = f^{(n)}(x) = \underset{h=0}{\mathcal{L}} \frac{\delta_h^n f(x)}{\delta_h^n x},$$

as distinguished from the notations of the Differential Calculus,

$$\left[\frac{d^n}{dx^n}\right] f(x) = f^n(x) = \underset{h=0}{\mathcal{L}} \frac{\Delta_h^n f(x)}{h^n}.$$

3. The law of the distribution of the argument x_{g+rh} is perfectly arbitrary, except that x_{g+rh} must converge to the limit x_g as h converges to zero. In general, for an assigned law of argument distribution we shall have a Differell Calculus in which the differell operator is not repetitive, and for a different argument distribution law we shall have a different Differell Calculus. In each of these calculi any differell operator may be taken as an operator *per se,* as the fundamental operation of a new calculus in which the operations are repetitive. The first differell in any such Differell Calculus is the derivative of the function. For

$$\underset{h=0}{\mathcal{L}} \frac{\delta f(x_g)}{\delta x_g} = \underset{h=0}{\mathcal{L}} \frac{f(x_g) - f(x_{g+h})}{x_g - x_{g+h}} = f'(x_g).$$

4. Two general distribution laws suggest themselves at once,

$$x_{g+rh} = x_g + \varphi(r, h),$$
$$x_{g+rh} = x_g\, \psi(r, h).$$

In which $\varphi(r, h)_{h=0} = 0$ and $\psi(r, h)_{h=0} = 1$.

We may call the former the Addition Calculus and the latter the Multiplication Calculus. This latter term has already been used by Dr. McClintock in his Calculus of Enlargement,* wherein, at the close, he makes mention of such a calculus.

Of the Addition Calculi, the simplest is given by the law

$$\varphi(r, h) = h.$$

All differells, after the first are infinite. The first exists, however, and taken as an operation *per se,* may be made the basis of a repetitive calculus, which is the Differential Calculus. Such is the case for the first differell of any Differell Calculus. As another illustration, let

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* American Journal of Mathematics, Vol. II, p. 156.

$$x_{g+rh} = x_g - C_{r,1}h + C_{r,2}h^2 - \ldots + (-1)^r h^r = x_g - 1 + (1-h)^r .$$

Here, evidently,

$$\delta_h^n x_g = h^n .$$

And it appears that the nth differell of this calculus is

$$f^{(n)}(x) = f'(x) + f''(x) + \ldots + f^n(x) .$$

Any rth differell may be taken as the fundamental operation of a repetitive calculus.

In like manner the simplest Multiplication Calculus is furnished by the law

$$x_{g+rh} = x_g(1-h)^r ,$$

or what is the same thing, for the results are the same,

$$x_{g+rh} = x_g e^{rh} .$$

Because this form of the calculus has been considered symbolically by Dr. McClintock, in the paper above referred to, we shall proceed to notice it particularly as an illustration of what may be developed in other calculi.

II. THE MULTIPLICATION CALCULUS.—DIFFERELLS

5. Let the law of distribution be

$$x_{g+rh} = x_g(1-h)^r$$

or as it may be written

$$\log x_{g+rh} - \log x_g = r \log(1-h) .$$

The interval between the logarithms of the arguments is log $(1-h)$. Otherwise, if $x_{g+rh} = x_g e^{rh}$, we have

$$\log x_{g+rh} - \log x_g = rh ,$$

and the interval between the logarithms of the argument is simply h

In the first case we have

$$\delta_h^n x_g = x_g h^n ,$$

$$\delta_h^n x_g^m = x_g^m [1 - (1-h)^m]^n .$$

On taking the ratio and passing to the limit, we obtain

$$\left[\frac{d^n}{d^n x}\right] x^m = m^n x^{m-1}.$$

In the second case, we have

$$\delta_h^n x_g = x_g (1 - e^h)^n,$$

$$\delta_h^n x_g^m = x_g^m (1 - e^{mh})^n,$$

and as before

$$\left[\frac{d^n}{d^n x}\right] x^m = m^n x^{m-1}.$$

6. From the nature of the formation of the nth generalized difference, we have

$$\left[\frac{\delta_h^n}{\delta_h^n x}\right] [\varphi(x) + \psi(x)] = \frac{\delta_h^n \varphi(x)}{\delta_h^n x} + \frac{\delta_h^n \psi(x)}{\delta_h^n x},$$

and on passing to the limit,

$$\left[\frac{d^n}{d^n x}\right] [\varphi(x) + \psi(x)] = \varphi^{(n)}(x) + \psi^{(n)}(x).$$

Or, the differell of a sum of functions is the sum of the differells of the functions.

7. Let $nh = k$, a constant, so when $h = 0$, $n = \infty$. Then, if $0 < q < 1$, we have at any point x_{g+qk} between x_g and x_{g+k},

$$x_{g+qk} - x_{g+qk+rh} = x_g e^{qk} (1 - e^{rh}),$$

and by taking h small enough we have the distribution of the argument as nearly uniform as we choose throughout the neighborhood of any point x_{g+qk} in the interval (x_g, x_{g+k}).

8. To examine the formation of the differells of functions, we have by Taylor's formula,

$$f(a + x) = \sum_0^\infty \frac{x^r}{r!} f^r(a).$$

The nth differell of this function with respect to x, is

$$f^{(n)}(a + x) = \sum_1^\infty r^n \frac{x^{r-1}}{r!} f^r(a),$$

a series which is convergent when Taylor's is, since they have the same ratio of convergency.

For $a + x$ substitute x, then

$$f^{(n)}(x) = \sum_1^\infty r^n \frac{(x-a)^{r-1}}{r!} f^r(a).$$

The next differell is

$$f^{(n+1)}(x) = \sum_1^\infty r^{n+1} \frac{(x-a)^{r-1}}{r!} f^r(a).$$

Multiply $f^{(n)}(x)$ by $(x - a)$ and differentiate, whence

$$f^{(n+1)}(x) = \frac{d}{dx}\left[(x-a) f^{(n)}(x)\right].$$

Again, multiply $f^{(n+1)}(x)$ by dx and integrate between x and a, whence results,

$$\int_a^x f^{(n+1)}(x)\, dx = (x-a) f^{(n)}(x).$$

Thus we find that $f^{(n)}(x)$ is the mean value of $f^{(n+1)}(x)$ between x and a. $f^{(n)}(x)$ is therefore a function of an interval, and has distinct reference to an argument interval, one end of which is a, called the base of reference of differells.

When the interval vanishes, we have, in general,

$$f^{(n+1)}(a) = f^{(n)}(a) = f'(a).$$

If zero is the base, we have correspondingly

$$f^{(n+1)}(x) = \frac{d}{dx}\left[x f^{(n)}(x)\right],$$

$$\int_0^x f^{(n+1)}(x)\, dx = x f^{(n)}(x).$$

9. We shall use the base a, unless specifically mentioned. By successive applications of the formulæ of the preceding article, we may express $f^{(n)}(x)$ in terms of the first n derivatives of $f(x)$.

We have, by definition,

$$f^{(1)}(x) = f'(x).$$

$$\begin{aligned}
\therefore\quad f^{(2)}(x) &= f'(x) + (x-a) f''(x),\\
f^{(3)}(x) &= f'(x) + 3(x-a) f''(x) + (x-a)^2 f'''(x),\\
f^{(4)}(x) &= f'(x) + 7(x-a) f''(x) + 6(x-a)^2 f'''(x) + (x-a)^3 f^{\text{iv}}(x),\\
f^{(5)}(x) &= f'(x) + 15(x-a) f''(x) + 25(x-a)^2 f'''(x) + 10(x-a)^3 f^{\text{iv}}(x)\\
&\quad + (x-a)^4 f^{\text{v}}(x),
\end{aligned}$$

and so on.

Putting

$$f^{(n)}(x) = \sum_1^n A_r (x - a)^{r-1} f^r(x) \, ,$$

$$f^{(n+1)}(x) = \sum_1^{n+1} A_r' (x - a)^{r-1} f^r(x) \, ,$$

we have for the law of formation

$$A_r' = r A_r + A_{r-1} \, ,$$

and

$$A_1 = 1 \, , \quad A_0 = A_{n+1} = 0 \, .$$

We may express this law more concisely in several ways, thus :*

$$(x - a) f^{(n)}(x) = \left[\frac{d}{dy}\right]^n f(x) = \left[(x-a)\frac{d}{dx}\right]^n f(x)$$

$$= \sum_1^n \Delta^r 0^n \, . \, \frac{(x - a)^r}{r!} f^r(x) \, ,$$

wherein $x - a = e^y$, and $\Delta^r 0^n$ is the well known number in the Finite Difference Calculus.

Also

$$f^{(n)}(x) = \left[1 + \frac{d}{dy}\right]^{n-1} f'(x) \, .$$

Again,

$$f^{(n+1)}(x) = \left[\frac{d}{dx}(x - a)\right]^n f'(x) \, ,$$

$$f^{(n+r)}(x) = \left[\frac{d}{dx}(x - a)\right]^r f^{(n)}(x) \, ,$$

in which the exponent of the square bracket indicates successive applications of the enclosed operator.

10. If we take the expressions of the above article as defining differells, we show that

$$\left[\frac{d^n}{d^n x}\right](x - a)^m = m^n (x - a)^{m-1},$$

where a is the base, as follows :—

$$f^{(2)}(x) = \frac{d}{dx}\left[(x - a) f'(x)\right] = \frac{d}{dx}\left[m (x - a)^m\right] = m^2 (x - a)^{m-1},$$

$$f^{(3)}(x) = \frac{d}{dx}\left[(x - a) \, m^2 (x - a)^{m-1}\right] = m^3 (x - a)^{m-1}.$$

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* Herschel's Theorem. See Laurent, Traité d'Analyse, Vol. III, p. 436.

If it is true for p, then

$$f^{(p+1)}(x) = \frac{d}{dx}\left[(x - a)\, m^p\, (x - a)^{m-1}\right] = m^{p+1}(x - a)^{m-1},$$

which establishes the result generally.

11. We can, by reversing the process of Art. 9, express $f^{(n)}(x)$ in terms of the first n differells, thus :—

$$\begin{aligned}(x - a) f''(x) &= f^{(2)}(x) - f^{(1)}(x)\,,\\ (x - a)^2 f'''(x) &= f^{(3)}(x) - 3f^{(2)}(x) + 2f^{(1)}(x)\,,\\ (x - a)^3 f^{\text{iv}}(x) &= f^{(4)}(x) - 6f^{(3)}(x) + 11f^{(2)}(x) - 6f^{(1)}(x)\,,\end{aligned}$$

and so on. Or, generally,

$$(-1)^n (x - a)^{n-1} f^n(x) = \sum_1^n A_r f^{(r)}(x)\,,$$

$$(-1)^{n+1} (x - a)^n f^{n+1}(x) = \sum_1^{n+1} A_r' f^{(r)}(x)\,,$$

wherein

$$A_r' = nA_r - A_{r-1}\,,$$

and

$$A_n = 1\,,\quad A_0 = A_{n+1} = 0\,.$$

Since, when $x = a$, $f^{(r)}(a) = f'(a)$, we must have $\sum_1^n A_r = 0$. The above results serve to define the nth derivative as

$$f^n(a) = \underset{x=a}{\pounds}\, \frac{\sum^n A_r f^{(r)}(x)}{(-1)^n (x - a)^{n-1}}\,.$$

The general law for the expression of the nth derivative in terms of the first n differells will be given in compact form later.

12. We have heretofore considered h as a positive convergent to zero ; when this is so and $0 < h < 1$, the arguments are distributed, we shall say progressively, from x_g in the order $x_{g+rh}, \ldots, x_{g+h}, x_g$ for the binomial form of distribution $x_g (1 - h)^r$, and in the order $x_g, x_{g+h}, \ldots, x_{g+rh}$ for the exponential form $x_g e^{rh}$. If h is a negative convergent to zero the distributions are regressive and both in the binomial and in the exponential form are in the order $x_{g+rh}, \ldots, x_{g+h}, x_g$. It is easily seen that the corresponding progressive and regressive differells are the same functions in either case.

13. *Fractional Differells.* Heretofore we have considered the order of a differell as an integer, but as these operations are not repetitive, there is nothing in the formation of these functions to thus restrict us.

Let ρ be any numerical ratio, then

$$\varepsilon_h^\rho x_g = x_{g+\rho h} \,,$$

$$\begin{aligned} \delta_h^\rho x_g &= (1 - \varepsilon_h)^\rho x_g \,, \\ &= (1 - C_{\rho,1}\,\varepsilon_h + C_{\rho\,2}\,\varepsilon_h^2 - \ldots)\,x_g \,. \end{aligned}$$

The expression of an operator as an infinite series is one so frequently made use of, that we shall use it here in the same sense as it is employed elsewhere in mathematical investigations.

$$\therefore \qquad \delta_h^\rho x_g = x_g - C_{\rho,1}\,x_{g+h} + C_{\rho,2}\,x_{g+2h} - \ldots .$$

This series must be convergent in order to give meaning to the results. If the law of distribution is $x_g\,(1 - h)^r$, and h is a positive number between zero and two, the series does converge, and we have

$$\begin{aligned} \delta_h^\rho x_g &= x_g\,[1 - (1 - h)]^\rho \,, \\ &= x_g h^\rho \,. \end{aligned}$$

In like manner, we have

$$\delta_h^\rho x_g^m = x_g^m\,[1 - (1 - h)^m]^\rho \,,$$

and there results from the limiting ratio,*

$$\left[\frac{d^\rho}{d^\rho x}\right] x^m = m^\rho x^{m-1} \,,$$

the same function as when ρ is an integer. When h is a negative convergent to zero, the infinite series above diverges, and we cannot say whether the regressive fractional derivative exists or not.

If, however, h is a negative convergent to zero and the law is $x_g e^{\rho h}$, the series converges and gives the same result as above, while in this case when h is a positive convergent the series is divergent.

In order to derive all the justification we can for these results, consider the law of distribution to be $x_g e^{\rho h}$. Now, n being integer, $\varepsilon_h^n x_g^m = x_{g+nh}^m$ for any value of m ; and

$$\delta_h^n x_g^m = (1 - \varepsilon_h)^n x_g^m \,.$$

But $\varepsilon_h^n x_g^m = (e^{mh})^n x_g^m$. Therefore, so far as the operation on x_g^m is concerned we may replace ε_h by e^{mh}. We have

$$\frac{\delta_h^\rho x_g^m}{\delta_h^\rho x_g} = \frac{(1 - \varepsilon_h)^\rho x_g^m}{(1 - \varepsilon_h)^\rho x_g} = \frac{(1 - e^{mh})^\rho x_g^m}{(1 - e^h)\,x_g} \,.$$

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* The base of operation here is zero.

Whether h be positive or negative, we have

$$\left[\frac{d^\rho}{d^\rho x}\right] x^m = m^\rho x^{m-1} .$$

Another symbolical verification of this result follows ; we have, in § 9,

$$f^{(\rho)}(x) = \left[1 + \frac{d}{dy}\right]^{\rho-1} f_x'(e^y) ,$$

when ρ is an integer ; assuming it true when ρ is fractional, let $f(x) = x^q$, then $f'(x) = qx^{q-1} = qe^{(q-1)y}$, and

$$\begin{aligned} f^{(\rho)}(x) &= (1 + C_{\rho-1,1} D_y + C_{\rho-1,2} D_y^2 + \ldots) qe^{(q-1)y} \\ &= qe^{(q-1)y} [1 + C_{\rho-1,1} (q-1) + C_{\rho-1,2} (q-1)^2 + \ldots] . \end{aligned}$$

If $-1 < q - 1 < +1$, the series converges, and

$$f^{(\rho)}(x) = q^\rho e^{(q-1)y} = q^\rho x^{q-1} .$$

For integral values of ρ we have the Differell Calculus as developed in the preceding articles. For each value of ρ fractional or integral and positive, we can develop a calculus in which the operative symbol is

$$\left[\frac{d^\rho}{d^\rho x}\right],$$

which may be taken as a repetitive operation.

Thus

$$\left[\frac{d^\rho}{d^\rho x}\right] \left[\frac{d^\rho}{d^\rho x}\right] \equiv \left[\frac{d^\rho}{d^\rho x}\right]^2 ,$$

and so on. In any such Rho-Differell Calculus, we have

$$\left[\frac{d^\rho}{d^\rho x}\right]^m x^q = q^\rho (q-1)^\rho \ldots (q - m + 1)^\rho x^{q-m}$$

which under certain conditions may be imaginary. This degenerates into the Differential Calculus when $\rho \equiv 1$.

14. *Fractional Derivatives.* The close relations existing between the Differell Calculus and the Differential Calculus, and the existence of the functions of the former when the operational index is fractional suggests an effort to define the operation upon the functions of the latter when the index of that operation is a fraction, through the existing relations of the operators in the two calculi. At the outset it must be seen that no fractional index of an

operation can be the result of a repetitive operation, but must indicate an operation *per se*, and upon the basis of this necessary truth rests the convention upon which depends the following interpretation.

Letting the base of differells be zero for convenience, we have, when n is a positive integer,

$$xf^{(n)}(x) = \left[\frac{d}{dy}\right]^n f(e^y),$$

wherein $x = e^y$. Let $f(x) = x^q$ (q being any number), then $f(e^y) = e^{qy}$, and we have from the Differell Calculus

$$xf^{(n)}(x) = xq^n x^{q-1} = q^n x^q.$$

From the Differential Calculus

$$\frac{d^n}{dy^n} e^{qy} = q^n e^{qy} = q^n x^q.$$

But when n is any positive fraction, say ρ, we have still

$$xf^{(\rho)}(x) = x\left[\frac{d^\rho}{d^\rho x}\right] x^q = q^\rho x^q.$$

And in the equalities

$$\left[\frac{d^\rho}{dy^\rho}\right] x^q = x\left[\frac{d^\rho}{d^\rho x}\right] x^q = q^\rho x^q,$$

we know that both equalities hold good for ρ a positive integer, and that the second equality is true for ρ any positive number. Let us assume that the first equality is also true when ρ is a positive fraction, thus defining the fractional operator in the symbol (d^ρ/dy^ρ). When ρ is an integer, the ordinary signification of

$$\left[\frac{d^\rho}{dy^\rho}\right] \equiv \left[\frac{d^\rho}{(dy)^\rho}\right] \equiv \left[\frac{d}{dy}\right]^\rho,$$

means that the operator d/dy is to be successively performed on a function ρ times, but when ρ is fractional no such repetitive meaning is possible for this operation and we can only interpret it as an operation *per se.*

The question arises, if

$$\left[\frac{d^\rho}{dy^\rho}\right] e^{qy} = q^\rho e^{qy},$$

is to be true for all positive values of ρ, what is the value of

$$\left[\frac{d^\rho}{dy^\rho}\right] y^q.$$

It is necessary that ρ should be a non-repetitive operator, whether integral or fractional, and considered as the symbol of an operation in itself. We have $x = e^y$ and $dy = dx/x$. Therefore

$$\left[\frac{d^\rho}{dy^\rho}\right] e^{qy} = \left[\frac{d^\rho}{(dx/x)^\rho}\right] x^q ,$$

which we write

$$\left[x^\rho \frac{d^\rho}{dx^\rho}\right] x^q ,$$

the value of which, is by the above definition $q^\rho e^{qy} = q^\rho x^q$.

When $\rho \equiv 1$, we have

$$\left[x \frac{d}{dx}\right] \equiv x \left[\frac{d}{dx}\right] .$$

We *assume*, inasmuch as ρ is no longer indicative of a repetitive operation, that

$$\left[x \frac{d}{dx}\right]^\rho \equiv \left[x^\rho \frac{d^\rho}{dx^\rho}\right] \equiv x^\rho \left[\frac{d^\rho}{dx^\rho}\right] .$$

Under this convention, we have

$$\left[\frac{d^\rho}{dx^\rho}\right] x^q = q^\rho \, x^{q-\rho} .$$

Otherwise, using the separation of symbols, as in the infinitesimal calculus, we have

$$\begin{aligned} d^\rho x^q &= q^\rho x^q \, [d\,(\log x)]^\rho \\ &= q^\rho x^q \left[\frac{dx}{x}\right]^\rho \\ &= q^\rho x^{q-\rho} (dx)^\rho . \end{aligned}$$

For any positive value of ρ, we can develop a calculus in which the fundamental operative symbol is

$$\left[\frac{d^\rho}{dx^\rho}\right] ,$$

which can be taken repeatedly furnishing the repetitive operation of this particular calculus. Thus

$$\left[\frac{d^\rho}{dx^\rho}\right] \left[\frac{d^\rho}{dx^\rho}\right] = \left[\frac{d^\rho}{dx^\rho}\right]^2$$

and so on. But the exponential law

$$\left[\frac{d^\rho}{dx^\rho}\right]^2 = \frac{d^{\rho+2}}{dx^{\rho+2}}$$

does not hold. In any such Rho-Differential Calculus, we have, assuming m to be an integer,

$$\left[\frac{d^\rho}{dx^\rho}\right]^m x^q = q^\rho (q-\rho)^\rho \ldots (q - \overline{m-1}\rho)^\rho x^{q-m\rho}\,,$$

which degenerates into the ordinary Differential Calculus when $\rho = 1$.

III. Expansion of Functions.

15. If the differell of a function vanishes for two values of the variable, the differell of next higher order must vanish for some value of the variable which lies between these two.

In virtue of $f^{(n)}(x)$ being the mean value of $f^{(n+1)}(x)$ between a and x, we have

$$f^{(n)}(x) = f^{(n+1)}(u)\,. \qquad a < u < x\,.$$

Now

$$\int_b^c f^{(n+1)}(x)\,dx = \int_a^c f^{(n+1)}(x)\,dx - \int_a^b f^{(n+1)}(x)\,dx$$

$$= (c-a)\,f^{(n)}(c) - (b-a)\,f^{(n)}(b)$$

$$= (c-b)\,f^{(n+1)}(u)\,. \qquad b < u < c\,.$$

If $f^{(n)}(c) = f^{(n)}(b)$,

$$\int_b^c f^{(n+1)}(x)\,dx = (c-b)\,f^{(n)}(c)\,.$$

In particular, if $f^{(n)}(c) = f^{(n)}(b) = 0$, then must

$$f^{(n+1)}(u) = 0\,, \qquad b < u < c\,.$$

It further follows from Art. 9 that, if the first n derivatives of $f(x)$ vanish at $x = b$, then must the first n differells vanish at $x = b$. Conversely, by Art. 11, if the first n differells vanish, or have equal values, at $x = b$, then must the first n derivatives vanish at $x = b$.

Again, if the first n derivatives or differells vanish at $x = b$, we have by Arts. 9, 11,

$$f^{(n+1)}(b) = (b-a)^n f^{n+1}(b)\,.$$

As x converges to the base a, then $f^{(n)}(x)$ converges to the value $f'(a)$.

16. Let $\varphi(x)$ and $\psi(x)$ be two functions which vanish together with their first n derivatives or differells, when $x = b$. Then

$$\mathop{\mathcal{L}}_{x=b} \frac{\varphi(x)}{\psi(x)} = \frac{\varphi^{n+1}(b)}{\psi^{n+1}(b)} = \frac{\varphi^{(n+1)}(b)}{\psi^{(n+1)}(b)}\,.$$

17. Let $F(x)$ and $F_1(x)$ be two functions which vanish, as well as their first n differells, for $x = c$. Then the function

$$J(x) = F_1(x_0)\,F(x) - F_1(x)\,F(x_0)$$

vanishes when $x = x_0$ and $x = c$, and also its first n differells and derivatives vanish for $x = c$.

Then must $J^{(1)}(x) = 0$, for $x = u_1$ between x_0 and c; and $J^{(2)}(x) = 0$, for $x = u_2$ between u_1 and c; and so on, until $J^{(n+1)}(x) = 0$ for $x = u$ between x_0 and c. The same may be said of the first $n + 1$ derivatives of $J(x)$. Hence we have

$$J^{(n+1)}(x) = F_1(x_0)\,F^{(n+1)}(u) - F_1^{(n+1)}(u)\,F(x_0) = 0\,.$$

If $F_1^{(n+1)}(u)$ or $F_1^{n+1}(u')$ is not zero between x_0 and c, we have, dropping the subscript zero,

$$\begin{aligned} F(x) &= F_1(x)\,\frac{F^{(n+1)}(u)}{F_1^{(n+1)}(u)} \\ &= F_1(x)\,\frac{F^{n+1}(u')}{F^{n+1}(u')}\,, \end{aligned}$$

wherein u and u' are values of x which lie somewhere between x and c, and therefore between a and c.

Now let

$$F_1(x) = (x - a)^{n+1}\,.$$

Then

$$\begin{aligned} F(x) &= \frac{x - a}{(n + 1)^{n+1}} \left[\frac{x - a}{u - a}\right]^n F^{(n+1)}(u) \\ &= \frac{(x - a)^{n+1}}{(n + 1)!}\,F^{n+1}(u')\,, \end{aligned}$$

for all values of x throughout the interval (ac). The interval (ac) being such that $a < c$, and u being confined to the interval $(x < c)$, the ratio

$$(x - a)/(u - a) < 1.$$

18. We now proceed to form a series which may be regarded as the fundamental series of this calculus, corresponding to the Taylor series of the Differential Calculus, and we shall derive it from the Generalized Difference Calculus in the same way that the Taylor series is derived from the ordinary Difference Calculus.

Remembering that

$$\delta_h^n = (1 - \varepsilon_h)^n\,, \quad \varepsilon_h^n = (1 - \delta\ )^n\,;$$

we have

$$\begin{aligned} f(x_{g+nh}) &= \varepsilon_h^n f(x_g) \\ &= (1-\delta_h)^n f(x_g) \\ &= f(x_g) - C_{n,1}\delta^1_h f(x_g) + C_{n,2}\delta^2_h f(x_g) - \dots + (-1)^n \delta_h^n f(x_g). \end{aligned}$$

Let the law of distribution be $x_{g+rh} = x_g(1-h)^r$, and $nh = k$, so that when $n = \infty$, $h = 0$. Then $x_{g+nh} = x_{g+k}$.

Since, $\delta_h^r x_g = x_g h^r$, we have

$$\begin{aligned} C_{n,r}\delta_h^r f(x_g) &= \frac{n(n-1)\dots(n-r+1)}{r!} x_g h^r \frac{\delta_h^r f(x_g)}{\delta_h^r x_g} \\ &= x_g (nh)^r \frac{\left[1-\frac{1}{n}\right]\dots\left[1-\frac{r+1}{n}\right]}{r!} \frac{\delta_h^r f(x_g)}{\delta_h^r x_g} \\ &= x_g \frac{k^r}{r!} f^r(x_g), \end{aligned}$$

when $n = \infty$. Also, we have,

$$\begin{aligned} x_{g+nh} = x_{g+k} &= x_g \left[1-\frac{k}{n}\right]^n, . \\ &= x_g e^{-k}, \end{aligned}$$

when $n = \infty$. Therefore, if convergent, we have the series

$$f(ce^{-k}) = f(c) + c\sum_1^\infty (-1)^r \frac{k^r}{r!} f^{(r)}(c),$$

wherein we have written c for x_g. Changing the sign of k, this becomes

$$f(ce^k) = f(c) + c\sum_1^\infty \frac{k^r}{r!} f^{(r)}(c).$$

Let $x = ce^k$, then $k = \log x - \log c$, and

$$f(x) = f(c) + c\sum_1^\infty \frac{(\log x - \log c)^r}{r!} f^{(r)}(c).$$

When the derivatives of $f(x)$ exist, we can write the coefficients in terms of the derivatives at c, § 9, and have*

$$f(x) = f(c) + c\sum_{r=1}^\infty \frac{(\log x - \log c)^r}{r!} \sum_{p=1}^r \Delta^p 0^r . \frac{(c-a)^p}{p!} f^p(c).$$

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* Herschel's theorem. See also Dr. McClintock's Calculus of Enlargement.

Putting $c = 1$, this series furnishes the means of defining the functions $f^{(r)}(c)$, as the coefficients of $(\log x)^r/r!$ in the expansion of $f(x)$ according to the ascending powers of the logarithm of the variable, as the derivatives have been defined as the coefficients in Taylor's series.

The remainder after n terms of this series is easily found, as follows:—

We have

$$\left[\frac{d}{dk}\right] f(ce^k) = x_{g+k}\, f^{(n)}(x_{g+k}),$$

and when $k = 0$, this becomes

$$\left[\frac{d}{dk}\right]_{k=0} f(c) = cf^{(n)}(c).$$

Let

$$F(k) = f(ce^k) - f(c) - c \overset{n}{\underset{1}{\Sigma'}} \frac{k^r}{r!} f^{(r)}(c).$$

This function vanishes when $k = 0$. Moreover if its first n derivatives are finite, uniform and continuous, they all vanish when $k = 0$, in virtue of the relation exhibited above.

The function

$$J(k) = k^{n+1} F(k_0) - k_0^{\,n+1} F(k)$$

vanishes when $k = k_0$, and the function and its n derivatives vanish when $k = 0$. Therefore, by § 17,

$$F(k_0) = \frac{k_0^{\,n+1}}{(n+1)!} F^{n+1}(u). \qquad 0 < u < k$$

Dropping the subscript, we have for the remainder of the series,

$$R = \frac{k^{n+1}}{(n+1)!} \left[\frac{d}{dk}\right]_{k=u}^{n+1} f(ce^u),$$

$$= \frac{k^{n+1}}{(n+1)!} u f^{(n+1)}(u),$$

$$= \frac{k^{n+1}}{(n+1)!} u \overset{n+1}{\underset{1}{\Sigma'}} \Delta^r 0^{n+1} . \frac{(u-a)^r}{r!} f^r(u).$$

19. The fundamental general theorem for the expansion of functions in the Differell Calculus, is this: The function

$$F(x) \equiv \frac{|\, f_0(x), f_1(x_g), f_2^{(1)}(x_g), \ldots, f_{n+1}^{(n)}(x_g) \,|}{|\, f_1(x_g), f_2^{(1)}(x_g), \ldots, f_{n+1}^{(n)}(x_g) \,|},$$

$$\equiv f_0(x) - \overset{n+1}{\underset{1}{\Sigma'}} A_r f_r(x),$$

wherein the A's, coefficients of $f_r(x)$, are independent of x but contain the differells of $f(x)$ at x_g, vanishes as do its first n differells when $x = x_g$. Let $F_1(x) = (x - x_g)^{n+1}$. Then by § 17,

$$F(x) = \frac{(x - x_g)^{n+1}}{(n+1)!} F^{n+1}(u) .$$

Or, putting c for x_g,

$$f(x) = \sum_1^{n+1} A_r f_r(x) + \frac{(x-c)^{n+1}}{(n+1)!} \left[f^{n+1}(u) - \sum_1^{n+1}{}' A_r f_r^{n+1}(u) \right] .$$

20. We give an independent proof of this theorem as follows. For brevity, let

$$\varepsilon_h^n f_m(x_g) = \varepsilon_m^n ; \quad \delta_h^n f_m(x_g) = \delta_m^n .$$

Consider the function

$$F(x) = \frac{\begin{vmatrix} f_0(x), & \dots, & f_{n+1}(x) \\ \varepsilon_0^0 \;, & \dots, & \varepsilon_{n+1}^0 \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ \varepsilon_0^n \;, & \dots, & \varepsilon_{n+1}^n \end{vmatrix}}{|\, \varepsilon_1^0, \ \varepsilon_2^1, \ \dots, \ \varepsilon_{n+1}^n \,|}$$

which vanishes when x takes each of the $n+1$ values x_{g+rh} $(r = 0, 1, 2, \dots, n)$. The value of this function is, by a theorem of the Differential Calculus (Annals Math., viii, 74),

$$F(x) = (x - x_g)(x - x_{g+h}) \dots (x - x_{g+nh}) \frac{F^{n+1}(u)}{(n+1)!} ,$$

wherein u lies between the greatest and least of the values x and x_{g+rh} $(r = 1, \dots, n)$.

Since we do not change the value of a determinant by subtracting rows from rows, we have

$$F(x) = \frac{\begin{vmatrix} f_0(x), & f_1(x), & \dots, & f_{n+1}(x) \\ \delta_0^0 \;, & \delta_1^0 \;, & \dots, & \delta_{n+1}^0 \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ \delta_0^n \;, & \delta_1^n \;, & \dots, & \delta_{n+1}^n \end{vmatrix}}{|\, \delta_1^0, \ \delta_2^1, \ \dots, \ \delta_{n+1}^n \,|} .$$

We still do not change the value of this function when we divide the row of

rth generalized differences in the numerator and denominator of the ratio by $\delta_h^r x_g$ for $r = 1, \ldots, n$, which expresses the terms in these rows as generalized difference-ratios. Letting h converge to zero we pass at once to the result of the previous article.

21. *Forms of Rational Integer Power Series.* Let $f_0(x) = f(x)$ and $f_r(x) = (x - a)^{r-1}$, then we have.

$$\frac{\begin{vmatrix} f(x) & 1, & (x-a), & \frac{(x-a)^2}{(c-a)}, & \ldots, & \frac{(x-a)^n}{(c-a)^{n-1}} \\ f(c) & 1, & c-a & , c-a & , \ldots, & c-a \\ f^{(1)}(c), & 0, & 1 & , 2^1 & , \ldots, & n^1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{(n)}(c), & 0, & 1 & , 2^n & , \ldots, & n^n \end{vmatrix}}{|\, 1,\ 2^2, \ldots, n^n \,|} = \frac{(x-c)^{n+1}}{(n+1)!} f^{n+1}(u),$$

wherein u lies between x and c. We have therefore

$$f(x) = A_0 + \sum_{p=1}^{n} (-1)^{p+1} A_p \frac{(x-a)^p}{(c-a)^{p-1}} + \frac{(x-c)^{n+1}}{(n+1)!} f^{n+1}(u),$$

wherein

$$A_p = \sum_{r=1}^{n} (-1)^{r+1} \frac{\Delta(r, p)}{\Delta} f^{(r)}(c).$$

$\Delta(r, p)$ is the determinant $\Delta = |\, 1^0, 2^2, \ldots, n^n \,|$, with its pth column and rth row deleted. The *remainder* of the series vanishes when $n = \infty$.

With the same notations as before, except now let

$$f_r(x) = \frac{1}{(r-1)!} \frac{(x-c)^{r-1}}{(c-a)^{r-2}},$$

we have

$$(x-a)\left[\frac{d^n}{d^n x}\right](x-c)^p = \sum_1^n \Delta^r 0^n \cdot \frac{(x-a)^r}{r!}\left[\frac{d}{dx}\right]^r (x-c)^p$$

$$= \sum_1^p \Delta^r 0^n \cdot C_{p,r}\,(x-a)^r (x-c)^{p-r}.$$

If $x = c$, then

$$\left[\frac{d}{d^n x}\right]^n_{x=c} (x-c)^p = \Delta^p 0^n \cdot (c-a)^{p-1}.$$

Moreover $\varDelta^p 0^n = 0$ for $p > n$ and is $p!$ when $n = p$. Therefore

$$\left|\begin{array}{cccccc} f(x), & 1, & \frac{1}{1!}\frac{(x-c)^1}{(c-a)^0}, & \frac{1}{2!}\frac{(x-c)^2}{(c-a)^1}, & \ldots, & \frac{1}{n!}\frac{(x-c)^n}{(c-a)^{n-1}} \\ f(c), & 1, & 0, & 0, & \ldots, & 0 \\ f^{(1)}(c), & 0, & 1, & 0, & \ldots, & 0 \\ f^{(2)}(c), & 0, & \frac{\varDelta^1 0^2}{1!}, & 1, & \ldots, & 0 \\ f^{(3)}(c), & 0, & \frac{\varDelta^1 0^3}{1!}, & \frac{\varDelta^2 0^3}{2!}, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{(n)}(c), & 0, & \frac{\varDelta^1 0^n}{1!}. & \frac{\varDelta^2 0^n}{2!}, & \ldots, & 1 \end{array}\right| = \frac{(x-c)^{n+1}}{(n+1)!} f^{(n+1)}(u).$$

The coefficients of $(x - c)^r$ are now in finite form. We are now prepared to give the general law promised in § 11, for the coefficient of $(x - c)^r$ in this series must equal the coefficient of the like power in Taylor's, therefore we have

$$(-1)^r (c-a)^{r-1} f^r(c) = \left|\begin{array}{ccccc} f^{(1)}(c), & 1, & 0, & \ldots, & 0 \\ f^{(2)}(c), & \frac{\varDelta^1 0^2}{1!}, & 1, & \ldots, & 0 \\ f^{(3)}(c), & \frac{\varDelta^1 0^3}{1!}, & \frac{\varDelta^2 0^3}{2!}, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{(r)}(c), & \frac{\varDelta^1 0^r}{1!}, & \frac{\varDelta^2 0^r}{2!}, & \ldots, & \frac{\varDelta^{r-1} 0^r}{(r-1)!} \end{array}\right|$$

If $c = a, f^{(r)}(c) = f'(a)$, and the determinant vanishes as was stated in § 11.

22. When $f(x)$ is a Taylor's function, we have, base zero,

$$\left[\frac{d^n}{d^n x}\right] f(x) = \sum_{r=1}^{\infty} r^n x^{r-1} \frac{f^r(0)}{r!} = \sum_{1}^{n} \varDelta^r 0^n \,.\, x^{r-1} \frac{f^r(x)}{r!}.$$

Put $f(x) = e^x$, then

$$e^x \sum_{1}^{n} \varDelta^r 0^n \,.\, \frac{x^{r-1}}{r!} = 1 + \frac{2^n}{2!} x + \frac{3^n}{3!} x^2 + \ldots$$

or the series on the right is the product of e^x and a rational integral. If $x = 1$,

$$e \sum_1^n \frac{\Delta^r 0^n}{r!} = 1 + \frac{2^n}{2!} + \frac{3^n}{3!} + \dots .$$

By assigning values to n we get different infinite series for e.

23. We have heretofore been using $f(x)$ as a Taylor's series. If the function cannot be expressed in Taylor's series, we must go back to the fundamental limiting ratio to determine the differells. For example, when the base is zero, $(\log x)^m$ cannot be expressed by Maclaurin's series. Its nth differell, however, is readily found to be

$$\left[\frac{d^n}{d^n x}\right] (\log x)^m = m(m-1)\dots(m-n+1) \frac{(\log x)^{m-n}}{x}.$$

For

$$\left[\frac{d^n}{d^n x}\right] (\log x)^m = \frac{1}{x}\left[x \frac{d}{dx}\right]^n (\log x)^m$$

$$= \frac{1}{x}\left[\frac{d}{d(\log x)}\right]^n (\log x)^m .$$

In like manner when the base is a, we have

$$\left[\frac{d^n}{d^n x}\right] [\log(x-a)]^m = \frac{1}{x-a}\left[\frac{d}{d\log(x-a)}\right]^n [\log(x-a)]^m$$

$$= \frac{m!}{n!}\frac{[\log(x-a)]^{m-n}}{x-a}.$$

We can now easily deduce the series of § 18, by the general method of §§ 19, 20.

Again, letting $f(x) = \log x$, we have

$$\sum_1^n (-1)^{r-1} \frac{\Delta^r 0^n}{r}$$

equal to unity for $n = 1$ and zero for $n = 1, 2, 3, \dots$.

IV. (*b*) INTEGRELLS OR NEGATIVE DIFFERELLS.*

24. Heretofore we have considered only positive values of n in using differells, we propose now to notice those corresponding functions in which n is negative. We shall, for brevity, call such functions *integrells*. The integrell bearing to the differell a relation somewhat analogous to that of the integral to the derivative.

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* **Read before The American Mathematical Society at Springfield, Mass., Aug., 1895.**

25. We might simply continue to define negative differells by using the property that each differell is the mean value of the one of next higher order. Whence, descending by units,

$$f^{(0)}(x) = \frac{1}{x-a}\int_a^x f^{(1)}(x)\,dx\,,$$

$$= \frac{f(x) - f(a)}{x-a}\,.$$

Which is the fundamental function with which this calculus deals. Geometrically pictured it represents the inclination of the secant through the points $[a, f(a)]$, $[x, f(x)]$ to the x-axis. This becomes the tangent to the curve $y = f(x)$ at $x = a$. Correspondingly

$$f^{(-1)}(x) = \frac{1}{x-a}\int_a^x f^{(0)}(x)\,dx\,,$$

is the mean inclination of the secant for all points in the interval (ax). Generally, we have

$$f^{(-n-1)}(x) = \frac{1}{x-a}\int_a^x f^{(-n)}(x)\,dx\,.$$

Or, the integrell of any order is the mean value of the integrell of next lower order. Clearly, if c is a constant,

$$\left[\frac{d^{-1}}{d^{-1}x}\right] c = 0\,.$$

If $f(x) = (x-a)^m$, it follows directly that

$$\left[\frac{d^{-n}}{d^{-n}x}\right](x-a)^m = \frac{(x-a)^{m-1}}{m^n}\,.$$

If zero be the base we have the corresponding values as for differells. Therefore, so far as Taylor functions are concerned the differells derived are true for positive and negative integral indices.

26. But, it is well to found the integrell as we did the differell, upon the limiting ratio. We have according to our original convention

$$\delta_h^{-n} \,.\, f(x_g) = (1-\varepsilon_h)^{-n} \,.\, f(x_g)$$

$$= \sum_{r=0}^{\infty} H_n^r \,.\, f(x_{g+rh})\,,$$

wherein $H_n^0 = 1$, and

$$H_n^r = \frac{1}{r!}\, n(n+1)\dots(n+r-1).$$

Since $(1-\varepsilon_h)^{-n}$ is expressed as an infinite series we must give this definite meaning before proceeding further. What we are to be concerned with, is the limiting ratio

$$\underset{h=0}{\pounds}\, \frac{\delta_h^{-n} f(x_g)}{\delta_h^{-n} x_g} = \underset{h=0}{\pounds}\, \frac{(1-\varepsilon_h)^{-n} f(x_g)}{(1-\varepsilon_h)^{-n} x_g}.$$

We have, when the distribution law is $x_{g-rh} = x_g e^{-rh}$,

$$\begin{aligned} \delta_h^{-n} x_g &= x_g \left[1 + ne^{-h} + \frac{1}{2!}\, n(n+1)\, e^{-2h} + \dots\right] \\ &= x_g\, (1 - e^{-h})^{-n}, \\ \delta_h^{-n} x_g^m &= x_g^m\, (1 - e^{-mh})^{-n}. \end{aligned}$$

And these are true whether n is integral or fractional. Consequently, we have

$$\begin{aligned} \left[\frac{d^{-n}}{d^{-n} x_g}\right] x_g^m &= \underset{h=0}{\pounds}\, \frac{\delta_h^{-n} x_g^m}{\delta_h^{-n} x_g} = x_g^{m-1}\, \underset{h=0}{\pounds} \left[\frac{1 - e^{-h}}{1 - e^{-mh}}\right]^n \\ &= \frac{x^{m-1}}{m^n}. \end{aligned}$$

But when we attempt to form the integrells of an absolute constant by this method, we are met by the difficulty that the result is infinity. For $\delta^n c = 0$, and

$$\varepsilon^n c = (1-\delta)^n c = c.$$

$$\therefore \qquad \delta^{-n} c = (1-\varepsilon)^{-n}\, c = c\,(1-1)^{-n} = \infty,$$

while

$$\delta^{-n} x = (1 - e^{-h})^{-n}.$$

Therefore we infer that the integrell operation applies only to functions which have no absolute term, that is to say to the difference of two values of a function.

We have

$$f(x+a) - f(a) = xf'(a) + \frac{x^2}{2!} f''(a) + \dots,$$

and

$$\left[\frac{d^{-n}}{d^{-n} x}\right] [f(x+a) - f(a)] = f'(a) + \frac{x}{2!\, 2^n} f''(a) + \frac{x^2}{3!\, 3^n} f'''(a) + \dots.$$

Now if we put $n = 0$, we define the integrell of zero order to be

$$\left[\frac{d^{-0}}{d^{-0}x}\right][f(x+a)-f(a)] = f'(a) + \frac{x}{2!}f''(a) + \frac{x^2}{3!}f'''(a) + \ldots,$$

$$= \frac{f(x+a)-f(a)}{x}.$$

Or, changing $x + a$ into x,

$$\left[\frac{d^{-n}}{d^{-n}x}\right][f(x)-f(a)] = f'(a) + \frac{x-a}{2!\,2^n}f''(a) + \ldots$$

$$\left[\frac{d^{-0}}{d^{-0}x}\right][f(x)-f(a)] = \frac{f(x)-f(a)}{x-a},$$

which we agree to write $f^{(0)}(x)$, also the nth integrell of the above function we agree to write $f^{(-n)}(x)$. Integrating $f^{(-n)}(x)$ between a and x, we find

$$f^{(-n-1)}(x) = \frac{1}{x-a}\int_a^x f^{(-n)}(x)\,dx.$$

Therefore the integrell is the mean value, between x and a, of the integrell of next lower order, and each has distinct reference to an interval whose base is a. The integrell of a constant is thus zero. The foundation of the integrells as here developed is the same as that of the preceding article.

27. Applying the above results to the function $(x - a)^m$, base a, we have for the nth integrell

$$\frac{(x-a)^{m-1}}{m^n}.$$

In like manner when c is not the base a, we have when $f(x) = (x - c)^m$,

$$f(x) - f(a) = C_{m,1}(a-c)^{m-1}(x-a) + C_{m,2}(a-c)^{m-2}(x-a)^2 + \ldots + C_{m,m}(x-a)^m.$$

Therefore

$$f^{(-n)}(x) = C_{m,1}(a-c)^{m-1} + C_{m,2}(a-c)^{m-2}\frac{(x-a)}{2^n} + \ldots + C_{m,m}\frac{(x-a)^{m-1}}{m^n}.$$

In this put $x = c$, then

$$f^{(-n)}(c) = m(a-c)^{m-1}\left[1 - \frac{C_{m-1,1}}{2^{n+1}} + \ldots + (-1)^{m-1}\frac{C_{m-1,m-1}}{m^{n+1}}\right].$$

Now we have (Chrystal's Algebra ii, 183),

$$\frac{n!}{x(x+1)\ldots(x+n)}P_p^n = \sum_{r=0}^{n}(-1)^r\frac{C_{n,r}}{(x+r)^{p+1}}.$$

Let $x = 1$, $n = m - 1$, $p = n$, and we have

$$1 - \frac{C_{m-1,1}}{2^{n+1}} + \ldots + (-1)^{m-1}\frac{C_{m-1,m-1}}{m^{n+1}} = \frac{1}{m} P_n^m,.$$

wherein P_n^m is the sum of the products of the numbers

$$\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{m}$$

taken n at a time with repetition. Therefore

$$f^{(-n)}(c) = (a - c)^{m-1} P_n^m .$$

28. Expansion of Functions:

We have

$$\varepsilon_h^{-n} = (1 - \delta_h)^{-n},$$

therefore

$$\varepsilon_h^{-n} f(x_g) = f(x_g) + H_n^1 \delta_h^1 f(x_g) + H_n^2 \delta_h^2 f(x_g) + \ldots$$

or,

$$f(x_{g-nh}) = f(x_g) + x_g \left\{ (nh) \frac{\delta^1 f}{\delta^1 x} + \frac{(nh)^2}{2!} \left[1 + \frac{1}{n}\right] \frac{\delta^2 f}{\delta^2 x} + \ldots \right\},$$

wherein $x_{g-rh} = x_g(1 - h)^r$. Let $nh = k$, and h converge to zero. Then

$$f(x_{g-k}) = f(x_g) + x_g \left\{ k f^{(1)}(x_g) + \frac{k^2}{2!} f^{(2)}(x_g) + \ldots \right\}$$

which is the same logarithmic expansion as was deduced in § 18.

29. If $F(x)$ be a uniform, finite, and continuous function from a to c, and if the first n integrells of $F(x)$ vanish at $x = c$, then must the function $F(x)$ have $n + 1$ distinct equal values between a and c, including these limits. We have

$$F^{(-n)}(c) = \frac{1}{c - a}\int_a^c F^{(-n+1)}(x)\, dx = 0 ,$$

Consequently

$$F^{(-n+1)}(u_1) = 0 . \qquad a < u_1 < c$$

But since

$$\int_a^c = \int_a^{u_1} + \int_{u_1}^c,$$

then must

$$F^{(-n+2)}(u_2) = 0 ; \quad F^{(-n+2)}(u_3) = 0 ,$$

$(a < u_2 < u_1)$, $(u_1 < u_3 < c)$. In like manner must $F^{(-n+3)}(x) = 0$, between

u_2 and a, u_2 and u_3, u_3 and c, and also at c. Continuing thus we see that $F^{(-n+r)}(x)$ must vanish for $(r + 1)$ distinct values of x between a and c, including c. Therefore $F^{(-1)}(x) = 0$, for n distinct values of x between a and c, including c. But

$$F^{(-1)}(x) = \frac{1}{x - a} \int_a^x \frac{F(x) - F(a)}{x - a}\, dx\,.$$

It follows that $F(x) = F(a)$ for these n values of x. Consequently $F(x)$ has $n + 1$ distinct equal values between a and c, inclusive.

30. If $F(x)$ be a Taylor's function in the interval (a, c), it has been shown in the Differential Calculus (Annals Math., viii, 74), that when it has $n + 1$ equal values in the interval, we have.

$$F(x) - F(c) = (x - c)\,(x - u_1) \ldots (x - u_{n-1})\,(x - a)\,\frac{F^{n+1}(u)}{(n + 1)!}\,,$$

wherein u lies between the greatest and least of a, c, u_1, ..., u_{n-1}, or in the interval (ac). We may write this result

$$F(x) - F(c) = \theta^{n+1}\,\frac{(a - c)^{n+1}}{(n + 1)!}\,F^{n+1}(u)$$

wherein $0 < \theta < 1$.

31. THEOREM.

$$F(x) = \frac{\begin{vmatrix} f(x) & , f_1(x) & , \ldots, & f_{n+1}(x) \\ f(c) & , f_1(c) & , \ldots, & f_{n+1}(c) \\ f^{(-1)}(c), & f_1^{(-1)}(c), & \ldots, & f_{n+1}^{(-1)}(c) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ f^{(-n)}(c), & f_1^{(-n)}(c), & \ldots, & f_{n+1}^{(-n)}(c) \end{vmatrix}}{[\text{minor of } f(x)]}$$

is such that its first n integrells and the function vanish at $x = c$. Therefore the function vanishes $n + 1$ times between a and c, inclusive, and by § 29, we have

$$F(x) = \theta^{n+1}\,\frac{(c - a)^{n+1}}{(n + 1)!}\,F^{n+1}(u)\,.$$

Moreover, since the function also vanishes at $x = a$, we can replace c by a in the second row of the numerator.

32. Let $f_{r+1}(x) = (x - a)^r$, then $F^{n+1}(u) = f^{n+1}(u)$, and

$$\frac{\begin{vmatrix} \dfrac{f(x) - f(a)}{x - a}, & 1, & \dfrac{x-a}{c-a}, & \ldots, & \left[\dfrac{x-a}{c-a}\right]^{n-1} \\ f^{(-1)}(c) & , 1, & 2^{-1} & , \ldots, & n^{-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{(-n)}(c) & , 1, & 2^{-n} & , \ldots, & n^{-n} \end{vmatrix}}{|\, 1^0,\ 2^{-2},\ \ldots,\ n^{-n} \,|} = \theta^{n+1}\frac{(c-a)^{n+1}}{(n+1!)} f^{n+1}(u).$$

Again, let $f_{r+1}(x) = (x - c)^r$, then as before $F^{n+1}(u) = f^{n+1}(u)$ and

$$\frac{\begin{vmatrix} \dfrac{f(x) - f(c)}{x - c}, & 1, & \dfrac{x-c}{a-c}, & \ldots, & \left[\dfrac{x-c}{a-c}\right]^{n-1} \\ f^{(-1)}(c) & , 1, & P_1^1 & , \ldots, & P_1^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{(-n)}(c) & , 1, & P_1^n & , \ldots, & P_n^n \end{vmatrix}}{|\, 1,\ P_2^2,\ \ldots,\ P_n^n \,|} = \theta^{n+1}\frac{(c-a)^{n+1}}{(n+1)!} f^{n+1}(u).$$

These series cannot be written in practicable form until the ratio $\Delta(r, p)/\Delta$ can be evaluated when $n = \infty$, as was the case in § 21.

V. Generalized Difference Ratios.

33. In forming the so-called generalized differences of a function, we departed from the ordinary method by changing the distribution of the argument, but retained the method of operation of differencing by forming the successive differences in the ordinary manner. We might have, more generally still, defined the nth generalized difference of a function at x_g to have been

$$\delta_h^n f(x_g) = \begin{vmatrix} f(x_g) & , 1, & p_0^1, & \ldots, & p_0^{n-1} \\ f(x_{g+h}) & , 1, & p_1^1, & \ldots, & p_1^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f(x_{g+nh}), & 1, & p_n^1, & \ldots, & p_n^{n-1} \end{vmatrix} \div |\, p_1^0,\ p_2^1,\ \ldots,\ p_n^{n-1} \,|,$$

wherein the p's are numbers ($p_0 = 0$), the law of whose formation is arbitrary. The corresponding difference of the argument is obtained by deleting the f. These differences evidently vanish when $h = 0$.

We have by expansion

$$\delta_h^n f(x_g) = \sum_0^n (-1)^r A_r f(x_{g+rh}),$$

wherein $A_0 = 1$, and

$$A_r = \frac{(p_n - p_0)\ldots(p_1 - p_0)}{(p_n - p_r)\ldots(p_{r+1} - p_r)(p_r - p_{r-1})\ldots(p_r - p_0)},$$

with a corresponding value for $\delta_h^n x_g$. We are concerned with the limiting ratio

$$\mathop{\pounds}_{h=0} \frac{\delta_h^n f(x_g)}{\delta_h^n x_g}.$$

If the law of the p's is $p_r = r$, we have the results of § 2.

34. Let the law of the p's be not $p_r = r$, and let the law of distribution be

$$x_{g+rh} = x_g + rh.$$

Then evidently

$$\mathop{\pounds}_{h=0} \frac{\delta_h^n f(x_g)}{\delta_h^n x_g} = \mathop{\pounds}_{h=0} \frac{\sum_0^n (-1)^r A_r f(x_g + rh)}{h \sum_1^n (-1)^r A_r},$$

$$= \mathop{\pounds}_{h=0} \frac{\sum_1^n (-1)^r r A_r f'(x_{g+rh})}{\sum_1^n (-1)^r r A_r},$$

$$= f'(x_g).$$

And generally

$$f^{m+1}(x_g) = \mathop{\pounds}_{h=0} \frac{\delta_h^n f^m(x_g)}{\delta_h^n x_g},$$

whatever be the value of the finite positive integer n.

If the law of the p's is not $p_r = \varphi(r)$, the same results hold when

$$x_{g+rh} = x_g + h\,\varphi(r),$$

as for example $\varphi(r) = 1/r$.

35. If the law is $x_{g+rh} = x_g e^{p_r h}$, we have the Calculus of Multiplication, as is easily verified by

$$\mathop{\pounds}_{h=0} \frac{\delta_h^n x^m}{\delta_h^n x} = m^n x^{m-1}.$$